III. *A general Investigation of the Nature of the Curve, formed by the Shadow of a prolate Spheroid, upon a Plane standing at right Angles to the Axis of the Shadow; in a Letter to the Royal Society, by Mr.* George Witchell, *F. R. S.*

Gentlemen,

I Beg leave to lay before the Royal Society the following investigation of an irregularity in the duration of the eclipses of Jupiter's satellites, occasioned by the figure of his body.

It has been known for a long time, that Jupiter's body was not truly spherical, but a prolate spheroid, and that in a much greater degree than any of the other planets; but notwithstanding this, it was never suspected that it would affect the durations of the eclipses of the satellite, till Dr. Bevis first thought of it, in the latter end of the summer 1761.

The Doctor, being at that time indisposed, recommended the subject to my consideration; and, in consequence of his request, I not long after presented him with a solution of the problem, being in substance

5

fubftance the fame with this, as far as propofition V. a copy of which he foon after tranfmitted to that excellent mathematician the late M. Clairaut.

In March 1763, M. de la Lande, an eminent French aftronomer, being here, Dr. Bevis fhewed him my paper; this occafioned a new article in the *Conn. des Mouv. Celeft.* 1765, p. 177, under the title, *Inégalité dans les demi-durées des éclipfes des fatellites de Jupiter, caufée par l'applatiffement de Jupiter:* in which he mentions this circumftance in the following words;
" M. le doɛteur Bevis me fit voir à Londres, au mois
" de Mars dernier, une folution rigoureufe & al-
" gebraïque de ce probleme, qui confifte à trouver la
" courbe qui réfulte de la feɛtion de l'ombre d'un
" fpheroïde à une diftance quelconque."

In this ftate it remained ever fince; for though the Doɛtor, and fome other gentlemen, to whom I fhewed it, frequently urged me to lay it before the Royal Society; I always declined it, till I fhould have time to make fome farther additions to it.

A few months fince, M. Bailly, a French gentleman, publifhed at Paris an elaborate treatife upon the theory of Jupiter's fatellites; in which he has been pleafed to give the honour of this difcovery intirely to M. de la Lande, without the leaft mention of Dr. Bevis. I then thought it incumbent on me to do juftice to the Doɛtor, by immediately finifhing my paper in the beft manner I was able, and prefenting it to the Royal Society.

I fhall be extremely glad, if this rude effay fhould excite fome more able perfon to treat the fubjeɛt in the manner it deferves; for though, I believe, my folution will not be deficient in point of truth, I am

not

not vain enough to think it may not be performed in a more elegant manner. I have the honour to be,

Gentlemen,

Your moſt obedient

humble ſervant,

Fleet-Street,
Jan. 7, 1767.
George Witchell.

L E M M A.

If any ſpheroid is cut by a plane, in any direction whatever (excepting that which is perpendicular to its axis), the figure of the ſection will be an ellipſis. This is demonſtrated in Simpſon's Fluxions, Vol. II. p. 456.

P R O P O S I T I O N I.

TAB. III. fig. 1. Let the ſphere BEGK be cut through its center by the planes BGK, BPD, BoD, BOD, EAK, and LPH; it is required to determine the inclination of the planes LPH, BOD, and alſo the inclination of the right lines AC, BC, which is meaſured by the arc AB; there being given the angles of inclination EBF, FBa, together with the arc BF: the angles AFB, EAL, being right angles, and the inclination of the required plane BOD, but little exceeding that of the given plane BoD.
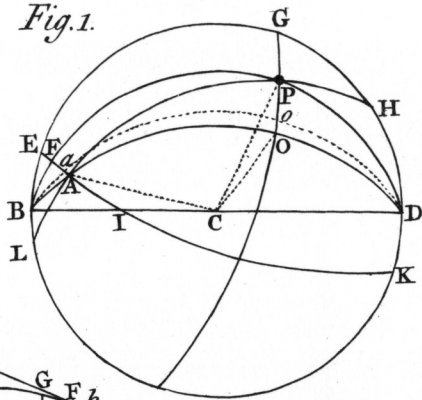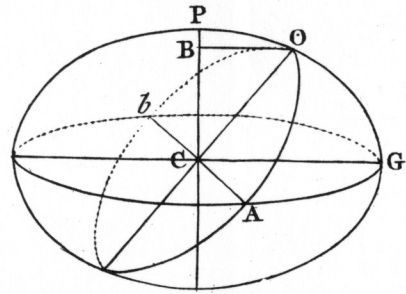
Let

Fig. 1.

Fig. 2.

Fig. 3.
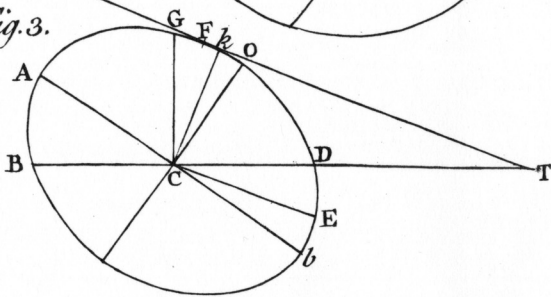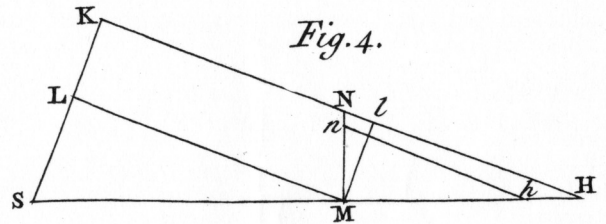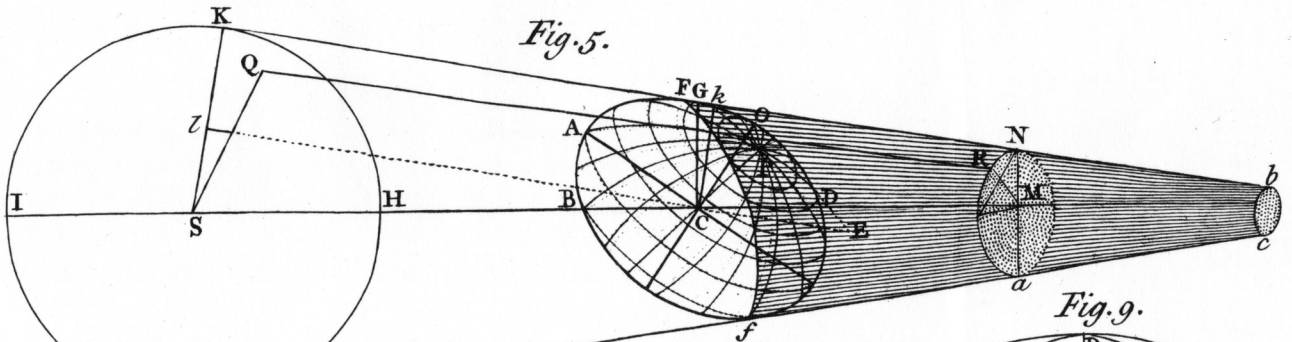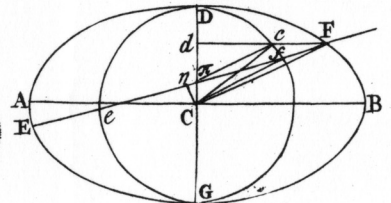
Fig. 4.

Fig. 5.

Fig. 6.

Fig. 7.

Fig. 8.
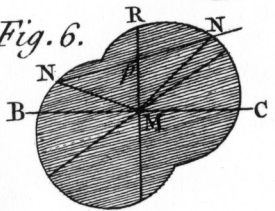
Fig. 9.

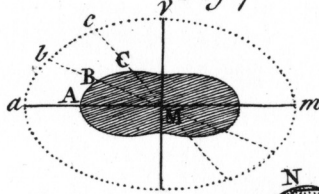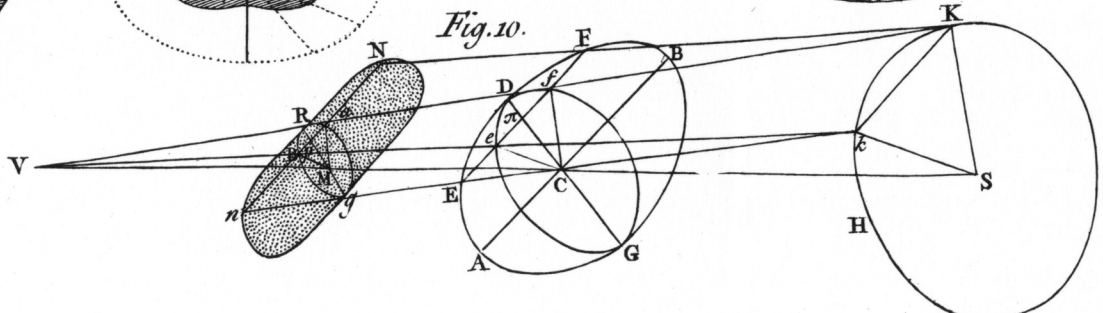Fig. 10.

Let the fine of EBF $=a$, its cofine $=a'$, the tangent of BF $=\beta$, its cofine $=b'$, the fine of FB$a=p$, its co-fine $=p'$, the fine of aBA $=z$, the fine of AB $=Z$, its cofine $=\acute{Z}$, the fine of PAO $=\zeta$, and radius $=1$; then we fhall have the fine of ABF $=$ the fine of $\overline{\text{FB}a+a\text{BA}}=p+p'z$, and its cofine $=p'-pz$: Therefore by trigonometry we fhall have in the right angled fpherical triangle ABF, as Rad. (1) : cofine BF (b') :: fine ABF $(p+p'z$: cofine BAF $=$ fine of LAB, or its equal PAO; therefore $\zeta=b' \times \overline{p+p'z}=$ the fine of the required inclination of the planes LPH, BOD. In like manner in the fame triangle it will be as rad. (1) : cotan. BF $\left(\frac{1}{\beta}\right)$ :: cofine ABF $(p'-pz)$: cotan. BA $=\frac{\acute{z}}{z}$;

hence $Z=\dfrac{\beta}{\sqrt{\beta^2+\overline{p'-pz}|^2}}$, and $\acute{Z}=\dfrac{p'-pz}{\sqrt{\beta^2+\overline{p'-pz}|^2}}$, which are the fine and cofine of the required arc AB.

COROLLARY I.

If inftead of a fphere we now fuppofe BEGK reprefents a prolate fpheroid, whofe axis is CP; the figures of the fections LPH, BOD, &c. inftead of circles, will become ellipfes (by the lemma); but it is evident that the inclinations of thofe planes to each other, and likewife the inclination of the right lines AC, BC, or the angle ACB, will remain unaltered.

COROLLARY II.

If BEGK reprefents any primary planet revolving about the fun in an orbit whofe plane co-incides
with

with the plane BCD, it is manifeſt that BCD will be its ecliptic, making the angle of obliquity BIE with its equator EAK (whoſe pole is P); and if B be the place of the ſun in this ecliptic, at any given time, the arc BI will be the diſtance of the ſun from the neareſt equinoctial point I; and the arc BF his declination at the ſame time.

COROLLARY III.

If the plane POG, which paſſes through P, the pole of the ſpheroid, be perpendicular to the plane LPH, it will alſo be perpendicular to any other plane BOD, which paſſes through A, the interſection of the equatorial plane EAK, with the plane LPH; therefore the angle ACO being a right angle, it is evident that AC will be the ſemi-tranſverſe, and CO the ſemi-conjugate axis of the elliptic ſection BOD.

COROLLARY IV.

Hence it appears, that the tranſverſe axis of any elliptic ſection BOD, made by a plane paſſing through the center of the ſpheroid, will always be equal to the equatorial diameter of the ſpheroid, but the conjugate axis will be longer or ſhorter, according as the inclination of the planes LPH, BOD, is more or leſs.

PROPOSITION II.

FIG. 2. To find the length of the ſemi-conjugate axis CO, of the elliptic ſection AO*b*, formed by a plane cutting the given prolate ſpheroid POG through its center C, and making the angle PCO with the axis CP.

Let

Let the fine of the angle $PCO = \zeta$, $CG = t$, $CP = c$, $CO = x$, (radius being unity); draw PO perpendicular to CP: then in the right angled plane triangle BCO, we have as rad. $(1) : CO \ (x) :: $ fine $PCO \ (\zeta) : BO \ (x\zeta)$; and rad. $(1) : CO \ (x)$ :: cofine $PCO \ (\sqrt{1-\zeta^2}) : BC \ (x\sqrt{1-\zeta^2})$; but from the nature of the ellipfis we have $\dfrac{c^2}{t^2} \times \overline{t^2 - x^2 \zeta^2}$

$= \overline{BC}|^2 = x^2 - x^2 \zeta^2$; therefore $x^2 = \dfrac{t^2 c^2}{t^2 - \overline{t^2 - c^2} \times \zeta^2}$,

or putting $t^2 - c^2 = f^2$, and $t^2 - x^2 = \varphi^2$, we have $x^2 = \dfrac{t^2 c^2}{t^2 - f^2 \zeta^2}$, and $\varphi^2 = \dfrac{t^2 \times \overline{1 - \zeta^2}}{\frac{t^2}{f^2} - \zeta^2}$.

PROPOSITION III.

Fig. 3. Let BOD be an ellipfis, whofe tranfverfe diameter Ab makes the angle ACB, with the right line BCD, and let TkG be a tangent to the ellipfis, in the point F, making the angle GTC with the right line BCD: It is required to find the length of the normal Ck, drawn from the center of the ellipfis, to the tangent TG.

From C, the center of the ellipfis, let CE be drawn parallel to the tangent TG, meeting the ellipfis in the point E; and CG perpendicular to the line BCD, meeting the tangent in the point G: Put the fine of A C B $= Z$, its cofine $= \acute{Z}$, the fine of TGC $= V$, its cofine $= \acute{V}$ (radius being unity) AC $= t$, CO $= x$, and $t^2 - x^2 = \varphi^2$; then will the fine of O C E ($=$ the fine of O C D $+$ D C E)

be expreſſed by $Z V + Z \acute{V}$, and by the laſt propoſition we ſhall find $CE = \dfrac{t_x}{\sqrt{t^2 - \varphi^2 \times \overline{Z V + Z \acute{V}}}}$; but (by conics) $CE \times Ck = CO \times CA$, whence we ſhall obtain $Ck = \sqrt{t^2 - \varphi^2 \times \overline{Z V + Z \acute{V}}^2}$.

PROPOSITION IV.

Fig. 4. In the two ſimilar right angled plane triangles H K S, H M N, right angled at K, and M, there is given the right lines K S and M S, to find the acute angles, ſuppoſing the given angle bnM to be nearly equal to the required angle HNM. Put $MS = \Delta$, $KS = r$, $MN = v$, the ſine of the given angle bn M $= q$, its coſine $= q'$, the ſine of HNM $= V$, its coſine $= \acute{V}$, the ſine of $\overline{HNM - bn\,M} = x$, and radius $= 1$. Let M L be drawn parallel to H K, and Ml parallel to S K: then in the right angled plane triangles N M l, SML, we have as rad. (1) : MN (v) :: ſine HNM (V) : Ml (vV), and as rad. (1) : MS (Δ) : ſin. LMS $(\acute{V})$: LS $(\Delta\acute{V})$; but Ml + LS = KS; therefore $vV + \Delta\acute{V} = r$, and by the foregoing notation $V = q + q'x$, and $\acute{V} = q' - qx$; therefore theſe values of V and $\acute{V}$ being wrote in the above equation we ſhall find $x = \dfrac{q'\Delta + qv - r}{q\Delta - q'v}$, and from thence $V = \dfrac{\Delta - q'r}{q\Delta - q'v}$, and $\acute{V} = \dfrac{qr - v}{q\Delta - q'v}$.

PROPOSITION V.

Fig. 5. If the opake prolate fpheroid B P O D, given in fpecies and pofition, be oppofed to the given luminous fphere H K Q I at the given diftance C S, forming the fhadow F*fbc* : It is propofed to determine the figure of the fection *a* R N made by a plane, cutting the fhadow perpendicularly to its axis at the given diftance M S.

Let the required curve *a* R N be conceived to be generated by the extremity R, of the variable right line M R, revolving about the given point M as a center, the line M R being always perpendicular to the axis of the fhadow MS: Let the right line RQ be a tangent to the fphere HKQI in the point Q, and in the fame plane with the right lines R M, M S, it will then reprefent one of the rays of light, which conftitute the conical fuperficies of the fhadow, and, therefore, by the laws of optics, will be a tangent to the fpheroid alfo; now when the generating point R has arrived at N, the ray R Q (being fuppofed to revolve with it) will co-incide with the tangent N K, touching the fphere in K, and the fpheroid in F : Join K, S, and the angles N M S, and N K S, will be right angles; let the fpheroid be fuppofed to be cut, by the quadrangular plane N M S K, forming thereby the elliptic fection B O D, draw C *k* perpendicular, and C *l* parallel to N K; put C A = t, M C = δ, C S = d, MS = Δ, SK = r, MN = v, CO = x, the fine of *b* N M = V, its cofine = $\acute{V}$, the fine of A C B = Z, its cofine = Z, and radius = 1 :

Then

Then in the right angled plane triangle C*l*S, it will be as rad. (1) : C S (d) :: fine S C*l* $(\dot{V})$: S*l* $(d\dot{V})$, and confequently C*k* $(=$ K S $-$ S*l$) = r - d \dot{V}$; but by prop. III. C$k = \sqrt{r^2 - \varphi^2 \times \overline{zv + zv}^2}$, whence we fhall have $r - d\dot{V} = \sqrt{r^2 - \varphi^2 \times \overline{zv + zv}^2}$: Now by propofition I. we fhall find $\zeta = b' \times \overline{p + p'z}$, $Z = \frac{\beta}{\sqrt{\beta^2 + \overline{p' - pz}^2}}$, and $\dot{Z} = \frac{p' - pz}{\sqrt{\beta^2 + \overline{p' - pz}^2}}$; by prop. II. $\varkappa^2 = \frac{r^2 c^2}{r^2 - f^2 \zeta^2}$, and $\varphi^2 = \frac{r^2 \times \overline{1 - \zeta^2}}{\frac{r^2}{f^2} - \zeta^2}$; laftly by prop. IV.

$V = \frac{\Delta - q'r}{q\Delta - q'v}$, and $\dot{V} = \frac{qr - v}{q\Delta - q'v}$, which values being fubftituted in the above equation will exhibit the nature of the required curve *a*R N, in terms of $\varkappa$ and v.

S C H O L I U M.

If the fphere H K Q I reprefents the fun, and the fpheroid B P O D one of the primary planets, it will appear, from the preceding reafoning, that the figure of the fection of its fhadow received upon a plane, which is perpendicular to its axis, will not be a circle (except when the axis of the planet produced paffes through the fun's center) but a curve of the oval kind, whofe fpecies will be known from the foregoing equation.

If the fphere H K Q I had been regarded as a fpheroid in the above folution, it is eafy to fee that the foregoing procefs would have determined the

nature

nature of the required curve; but the figure of the fun is fo nearly fpherical, that it was not thought neceffary to embarrafs the folution with that confideration.

Hence the duration of an eclipfe of a given fatelles may be determined in the following manner: Let B R C (fig. 6.) be the fection of the fhadow, through which the fatelles paffes, N p N the path of the fatelles, making the given angle N p M, with the circle of latitude R p M ; B M C a part of the primary's orbit produced, and M p the given latitude of the fatelles at the time of the fyzygia ; the circle of latitude R p M is reprefented in fig. 1. by the primitive circle B E G D, and the angle R M N, by the fpherical angle E B A ; therefore the fine of R M N $=$ the fine of E B A $=$ the fine of $\overline{E\,B\,F\,+\,F\,B\,a\,+\,a\,B\,A}$ $= a\,p' + a'\,p + \overline{a'\,p'} - ap \times z$, and its cofine $= a'p' - ap - \overline{a\,p'} + a'p \times z$; which for the fake of brevity may be expreffed by y, and y' ; then putting M $p = n$, M N $= v$, the fine of M p N $= m$, its cofine $= m'$, and radius $= 1$; we fhall have the fine of M N P expreft by $m\,y' + m'y$; and therefore we fhall have in the plane triangle M p N, as the fin. MN p $(m\,y' + m\,y)$: M p (n) :: fine M p N (m) : M N (v) ; hence $v = \frac{m.n}{my' + ym'}$; from which, and the equation of the curve (determined above) $\frac{v\,y}{m} = p$ N, and confequently, the duration of the eclipfe will become known.

In prop. I. the fine of the angle ABF is expreffed by $p + p'z$, and its cofine by $p' - pz$, inftead of their true values $pz' + p'z$, and $p'z' - pz$; this was done

to

to render the following conclusions more simple than they otherwise would have been; and as the angle *a* B A is, by hypothesis, but small, its cosine will approach so near to the radius, as not to occasion any sensible error in the result; and the same may be observed with regard to what is advanced in prop. IV.

It remains now to apply, what has been investigated above, to the eclipses of Jupiter's satellites, and to examine whether the prolateness of his figure will have any sensible effect upon their durations; and this is become the mcre necessary, as that celebrated astronomer M. de la Lande (who candidly acknowledges, that he was excited to turn his thoughts upon this subject, from a cursory view of this paper, which was shewn him by Dr. Bevis *) does not seem to have considered the question, with that degree of attention which I think it demands.

But before this can be done with exactness, it will be necessary to have the inclination of Jupiter's axis, with respect to his ecliptic, and the place of his equinoxes determined by observation, neither of which I believe has yet been done with any degree of certainty ; I shall, therefore, proceed in this inquiry upon M. de la Lande's hypothesis, that Jupiter's axis is perpendicular to his orbit ; and perhaps this supposition is not so far distant from the truth, as to occasion any material error in the conclusion. It may also be remarked, that in the general equation given above, V̀ and V express the sine and cosine of the semiangle of the cone of Jupiter's shadow, but this angle can never exceed $3'$, and consequently we may very

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* Vid. Connoiff. des Mouv. Celeft. 1765, p. 177.

safely

fafely ufe the radius inftead of V wherever it occurs.

By this means the general equation will become $r - d \overset{\prime}{V} = \sqrt{r^2 - \varphi^2}$, or which is the fame $r - d \overset{\prime}{V} = x$, therefore $\overset{\prime}{V} = \frac{r-x}{d}$; but by prop. IV. $\overset{\prime}{V} = \frac{qr-v}{q\Delta-q'v}$, which, becaufe q is nearly equal to V, and $q'v$ very fmall with refpect to $q\Delta$, will become $\overset{\prime}{V} = \frac{r-v}{\Delta}$; therefore $\frac{r-x}{d} = \frac{r-v}{\Delta}$, from which we fhall find $x = \frac{\Delta x - \delta r}{d}$; and this equation is exactly the fame with that which would arife from confidering the fun as a circular, and Jupiter as an elliptic plane, limited by one of his meridians, and always parallel to the difk of the fun; which fuppofition, the immenfe diftance of Jupiter from the fun renders very allowable.

From this equation an eafy mechanical method may be derived of delineating the curve of the fhadow, at any given diftance from Jupiter, for as x denotes any femi-diameter of the elliptic fection of Jupiter's body, it is manifeft, that the term $\frac{\Delta}{d} \times x$, will exprefs the correfponding femi-diameter of a fimilar ellipfis, whofe axes are to thofe of Jupiter in the given ratio of Δ to d, and the term $\frac{\delta r}{d}$ is wholly given:

Therefore if *a r m* (fig. 7.) be fuch an ellipfis, and there be drawn through its center M any number of femi-diameters M *a*, M *b*, M *c*, &c. meeting the el-
lipfis

lipfis in a, b, c, &c. let a A, b B, c C, &c. be taken each equal to the given term $\frac{\delta r}{d}$, and the points A, B, C, &c. will be in the required curve.

It appears from confidering the nature of this curve, that it will have two cufps, one at each extremity of its leffer axis, which will approach toward each other, according as the diftance δ is augmented; therefore, if the diftance of the fection of the fhadow, from Jupiter's center, was taken, fuch that $\delta = \frac{dc}{r-c}$, the leffer axis of the curve would then vanifh, and the cufps meet in the center, and thereby form two diftinct fhadows (as reprefented in fig. 8); in confequence of which, if a fatelles, revolved at that diftance, it might fuffer a double eclipfe, at the fame conjunction, which remarkable phænomenon may alfo happen, at a lefs diftance from Jupiter, in fome circumftances.

I fhall now fhew how the duration of an eclipfe of a given fatelles may be determined independant of the equation of the curve; and this, perhaps, will be the more acceptable, as it will afford a practical rule, which may be applied, in every pofition of Jupiter's axis, with very little trouble. This may be done by the help of the following propofition.

PROPOSITION VI.

If a circle e D f G be defcribed about the conjugate axis G D, of a given ellipfis A D B G, and a right line E F be drawn, making the given angle F π D,

with

with the conjugate axis, and paffing through the given point π taken therein, it is propofed to determine the length of the fegments F f, E e, intercepted between the circumference of the circle, and the perimeter of the ellipfis.

From the point F, draw the right line F d parallel to the tranfverfe axis A B, meeting the conjugate GD in the point d, and the circle in c; draw the lines C F, C f, C c, and let $c\pi$ be joined : Then by conics we fhall have, as CB : C D :: tang. F π D : tang. $c\pi$ D, and in the right lined triangle C πc, it will be as C c (C D) : fin. C πc :: C π : fin. C $c\pi$, whence the angle c C π becomes known; but as CD : CB :: tang. c C π : tang. FC π; therefore FC π is known ; from which taking away the given angle fC π, there remains the angle FC f; confequently all the angles, in the right lined triangle fCF, together with the fide C f (CD), are known : we fhall therefore have, in the right lined triangle, F f C, as fin. fF C : C f :: fin. fCF : fF, one of the required fegments, and by a fimilar operation, the other fegment E e will be found, whence as ef is given, E F will become known.

COROLLARY I.

The required fegments F f, E e, will be found in the fame manner, when the given point π is not taken in one of the axes, but any where between ; but in that cafe, the point where the line EF interfeds the conjugate axis, muft be firft determined.

COROLLARY II.

If a perpendicular C n be let fall from C upon the line E F, the angle π C n will be given, to which,

adding F C π (found above) the angle F C n will be known; hence we fhall have the following analogy for determining Fn: As tang. fCn : tang. F Cn :: fn : F n.

Now let KkH (fig. 10.) reprefent the difk of the fun, and eDfG that of Jupiter, confidered as a circle, whofe diameter is equal to his axis D G, draw N$p$$n$, the path of the fatelles, making the given angle NpR, with a right light Rg drawn parallel to the diameter D G, and let $a b$ be the duration of the eclipfe, and V the apex of the fhadow in this hypothefis; join Va, Vb, and let the plane aVb be produced, till it meets the fun's difk in K and k, it will then interfect the difk of Jupiter in the line $f\pi e$, and the lines VK, Vk, will alfo touch the circumference of the circle eDfG, in the points e and f, draw the line SV, and it will be the axis of the fhadow, and confequently will pafs through C and M, the centers of Jupiter, and the fection of the fhadow; join aM, bM, fC, eC, and the triangles $a b$M, $e f$C, will be fimilar to each other, and, therefore, $a b$M being wholly given, $f e$C will likewife be known. Let ADBG be the elliptic fection of Jupiter's body, and produce $e \pi f$ both ways, till it meets the periphery of the ellipfis in the points E and F, draw KF, kE, and produce them till they meet with $a b$, produced both ways in N and n, then will N n be the required duration of the eclipfe in the true fhadow: Now the triangles KfF, KaN, being fimilar, as are alfo the triangles $k e$E, $k b n$, and the fegments Ff, eE, being given by the preceding propofition, the required fegments Na, $b n$, will alfo become known, for they will be to the former fegments in the given ratio of S M to S C.

It

It may be obferved, that this method is equally applicable, whether the axis of Jupiter is perpendicular to his orbit, or not; for if it is not, we can eafily find by propofition I and II. the fpecies and pofition of that elliptic fection of Jupiter's body, to which a right line connecting the centers of the fun and Jupiter is perpendicular; and this being obtained every thing elfe will remain as before.

As it would require more time, than I have to fpare at prefent, to enter into a particular inquiry concerning the alterations, which this irregularity in the fhadow will occafion, in the prefent theory of Jupiter's fatellites, I fhall conclude with obferving, that the errors in the femi-durations of their eclipfes, arifing from this caufe, may fometimes amount to 20″ in the firft; 50″ in the fecond; 2′ 19″ in the third; and 11′ 14″ in the fourth; which errors will, I believe, be deemed fufficiently large to merit the attention of aftronomers.

G. Witchell.

IV. *An*